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A refinement on spectral Mantel's theorem. (English. English summary)

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All graphs considered are simple, undirected, and have no isolated vertices. The spectral version of Mantel's theorem due to E. Nosal [*Eigenvalues of graphs*, master's thesis, Univ. Calgary, 1970] and V. S. Nikiforov [Linear Algebra Appl. **418** (2006), no. 1, 257–268; MR2257594; erratum, R. A. Brualdi, Linear Algebra Appl. **420** (2007), no. 2-3, 727–728; MR2278199] says that the spectral radius $\rho(G)$ of any triangle-free graph G of size m is at most $\sqrt{m}$, with equality holding if and only if G is a complete bipartite graph. A natural problem along these lines that has recently received a lot of attention is to find the extremal graphs that attain the maximum spectral radius among particular classes of graphs of a fixed size. From a series of papers due to H. Lin, B. Ning and B. Wu [Combin. Probab. Comput. **30** (2021), no. 2, 258–270; MR4225787], M. Zhai and J. Shu [Discrete Math. **345** (2022), no. 1, Paper No. 112630; MR4318316], Y. Li and Y. J. Peng [Electron. J. Combin. **29** (2022), no. 4, Paper No. 4.2; MR4497214], W. Sun and S. Li [Discrete Math. **346** (2023), no. 7, Paper No. 113440; MR4571210], and Z. Lou, L. Lu and X. Huang [Electron. J. Combin. **31** (2024), no. 1, Paper No. 1.58; MR4718222], for odd m, the extremal graphs are known for the family of non-bipartite graphs of size m and *odd girth* (i.e., the length of a shortest odd cycle) being at least $2\ell+1$, for each $\ell > 1$. For even m, a recent work of Li, L. H. Feng and Peng [Electron. J. Combin. **31** (2024), no. 1, Paper No. 1.52; MR4718216] characterized the extremal non-bipartite triangle-free graphs of size m, for all sufficiently large m; in the same work, Li, Feng and Peng posed as an open problem to characterize, for each odd $k \geq 7$, the extremal non-bipartite graphs with (fixed) even size m and odd girth at least k.

This paper solves the problem for all sufficiently large even m, specifically when $m \geq 15k$ for each odd $k \geq 7$. The structure of the extremal graph depends on the residue class of $m-k$ modulo 3. The authors introduce a spectral technique of "residual functions" to characterize the structure of the extremal graphs, and then use the edge-switching method and double-eigenvector technique to determine the extremal graphs when $m-k \equiv 0, 1, 2 \pmod 3$.

Just as the classical Turán's theorem generalizes Mantel's theorem, a spectral version of Turán's theorem due to Nikiforov [Combin. Probab. Comput. **11** (2002), no. 2, 179–189; MR1888908] generalizes the spectral Mantel's theorem, and says that, for $r \geq 3$, we have $\rho(G) \leq \sqrt{2m(1-1/r)}$ for any K_{r+1}-free graph G of size m, with equality if and only if G is a regular complete r-partite graph. The authors raise the question of finding an analogous characterization of the extremal graphs for the class of non-r-partite K_{r+1}-free graphs of a fixed size.

A minor typo: in the second line after equation (26), "G_0" should instead be "G_1".

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.